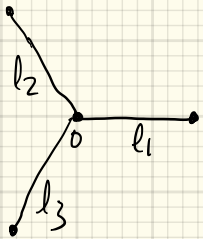
COUNTING TRIPODS (1)

ICERM Lattice Points
joint work with Aulicino, Delecroix, Neitzke
MSRI Fall 2019

Our goal is to count <u>tripods</u> of length $\leq R$ on the flat torus $\mathbb{C}/\mathbb{Z}[i]$.

<u>Def</u> A <u>tripod</u> consists of a non-zero pt. $p \in \mathbb{C}/\mathbb{Z}[i]$ w/ straight line segments emanating @ angles of $2\pi/3$ and ending @ 0. That is, it is an <u>immersed</u> copy of a metric graph $\mathcal{G}$ embedded in $\mathbb{C}$ together w/ an isometric immersion I to $\mathbb{C}/\mathbb{Z}[i]$

$\mathcal{G} =$ 

w/ $I(l_1) = I(l_2 e^{2\pi i/3})$

$I = (l_3 e^{4\pi i/3}) = 0$

$I(0) = p.$

(here isometric w.r.t. inherited metric on $\mathcal{G}$ & natural flat metric on $\mathbb{C}/\mathbb{Z}[i]$)

We denote the pair $(\mathcal{G}, I)$ by $\curlyvee$ & $\ell(\curlyvee) = \sum_{i=1}^{3} l_i$

COUNTING TRIPODS (2)

We would like to understand the $R \to \infty$ asymptotics of

$$N_{\text{prim}}(R) := \#\{\curlywedge : \ell(\curlywedge) \le R\}.$$

$\curlywedge$ is primitive.

In these notes, we will sketch a proof of the following

<u>Claim</u> $$N_{\text{prim.}}(R) \asymp \frac{1}{\zeta(4)} \frac{\sqrt{3}\pi}{24} \cdot R^4 = \frac{15\sqrt{3}}{4\pi^3} R^4$$

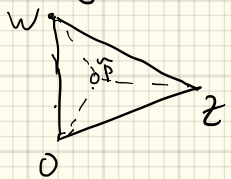
(where $\asymp$ means the ratio $\to 1$ as $R \to \infty$)

We will prove this claim by turning our problem into counting pairs $(z,w) \in \mathbb{Z}[i]^2$ satisfying certain conditions.

<u>Note</u>: $\curlywedge$ is <u>primitive</u> if it is not a scaled copy of another tripod.

COUNTING TRIPODS ③

Note that if we lift a (primitive) tripod from $\mathbb{C}/\mathbb{Z}[i]$ to $\mathbb{C}$ we obtain a center point $\tilde{p}$ & segments emanating from $\tilde{p}$ to pts. in $\mathbb{Z}[i]$. We can take one of these points to be 0, and we call the others z, w. We can remove (some) ambiguity from the situation by insisting that the largest angle of the triangle $\Delta(0zw)$ is @ 0 (i.e., $|w-z| \geq |z|, |w|$)



We can compute the <u>length</u> of the tripod $\chi(z,w)$ [only certain pairs z, w will yield a tripod]

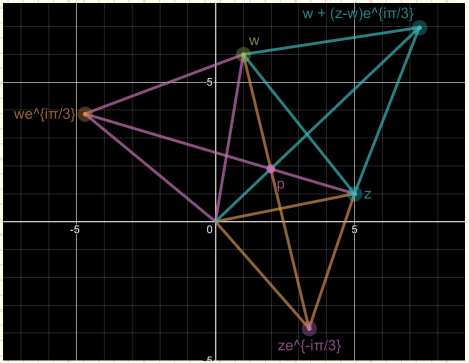
in terms of z, w.

COUNTING TRIPODS (4)

<u>Fermat points</u>: In fact, it is a beautiful fact from classical Euclidean geometry that a triangle has an inscribed tripod $\iff$ all angles are $\leq 2\pi/3$ (i.e., largest angle $\leq 2\pi/3$).

In this case, the center of the tripod can be constructed as follows: form an equilateral triangle on each side of the triangle, and connect the opposite vertex of this triangle to the opposite vertex of the original triangle.

- You can see this construction in the figure, w/ each side of the original triangle $\Delta(0zw)$ in a different color, & the equilateral triangles coded by the side they are built on. p is the tripod point.



- These lines intersect in the tripod point p, and $\ell(\succ)$ = length of any of these lines.

COUNTING TRIPODS (5)

Claim $\ell(\lambda(z,w)) = |w + (z-w)e^{i\pi/3}|$

$= |z - we^{i\pi/3}|$

$= |w - ze^{-i\pi/3}|$

(here, we use the colors in the figure to highlight the various lengths)

Pf. $\ell(\lambda(z,w)) = |p| + |w-p| + |z-p|$.

Note that $e^{i2\pi/3}(z-p) \parallel -e^{-i2\pi/3}p \parallel w-p$

so $\ell(\lambda(z,w)) = |e^{2\pi i/3}(z-p) + (w-p) - e^{-2\pi i/3}p|$

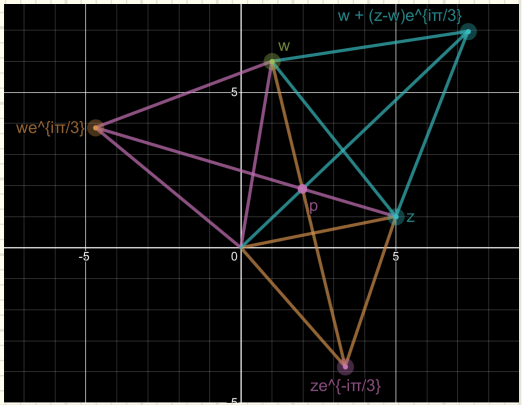
$= |ze^{2i\pi/3} + w + p(-e^{-i2\pi/3} - e^{i2\pi/3} - 1)|$

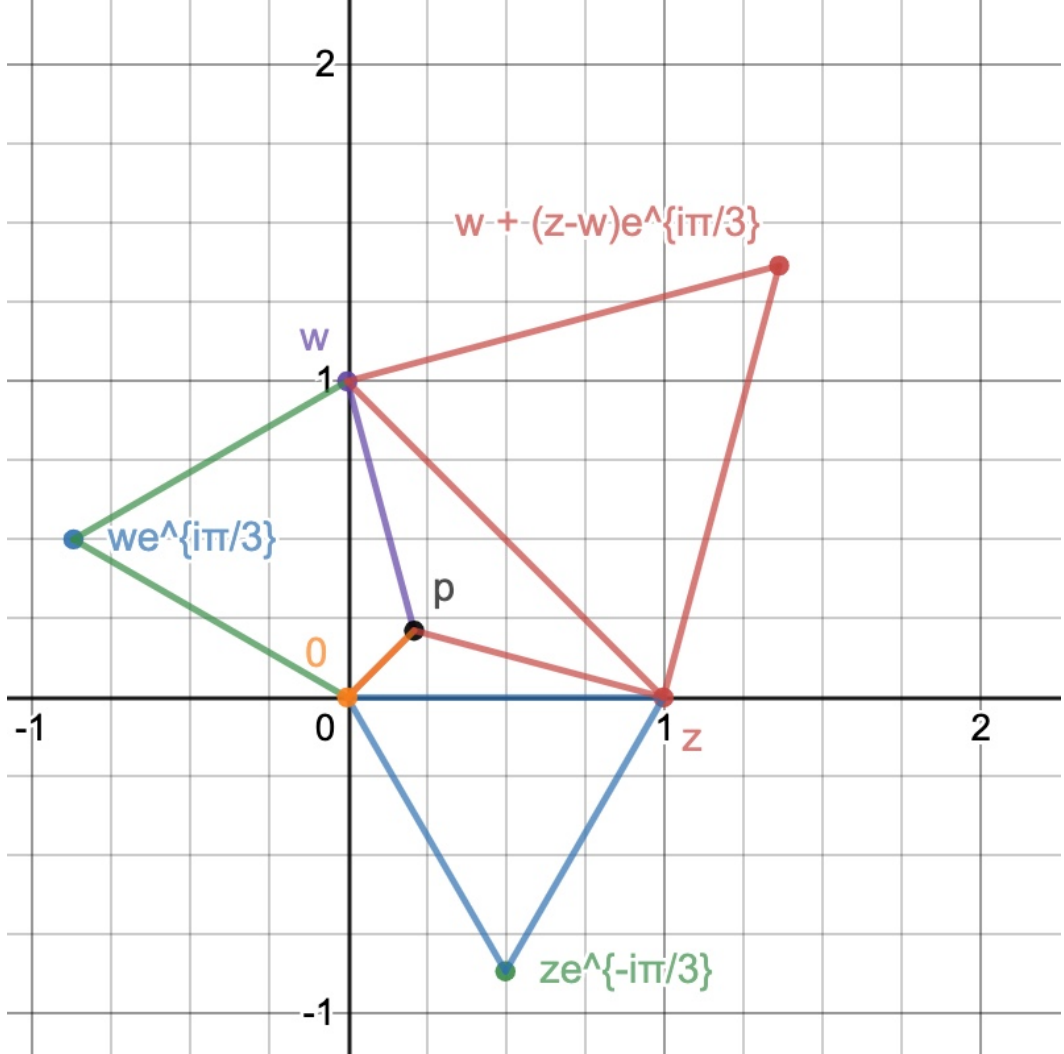
$= |ze^{2\pi i/3} + w| = |ze^{i\pi/3} + we^{-i\pi/3}|$

$= |w + (z-w)e^{i\pi/3}|$

$[\, e^{i2\pi/3} + e^{-i2\pi/3} = -1 \,]$

$[\, e^{i\pi/3} + e^{-i\pi/3} = 1 \,]$




w + (z-w)e^{iπ/3}
w
we^{iπ/3}
p
0
z
ze^{-iπ/3}
2
1
-1
0
1
2
-1

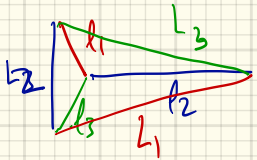
COUNTING TRIPODS (6)

DIGRESSION (1) The point p minimizes the function

$$f(\rho) = |\rho| + |z-\rho| + |w-\rho|$$

& is called the <u>Fermat</u> point of the triangle $\Delta(0zw)$. The resulting tripod is called the <u>Steiner tree</u>.

(2) One could consider different notions of tripod length.



$$\ell(\curlywedge) = l_1 + l_2 + l_3. \qquad \ell_2(\curlywedge) = \sqrt{l_1^2 + l_2^2 + l_3^2}$$

$$\ell_\Delta(\curlywedge) = L_1 + L_2 + L_3 \qquad \ell_{2,\Delta}(\curlywedge) = L_1^2 + L_2^2 + L_3^2$$

Law of cosines: $L_1^2 = l_2^2 + l_3^2 + l_2 l_3$, $L_2^2 = l_1^2 + l_3^2 + l_1 l_3$, $L_3^2 = l_1^2 + l_2^2 + l_1 l_2$

$[\, c^2 = a^2 + b^2 - 2ab\cos\theta, \ \cos 2\pi/3 = -1/2 \,]$.

So $$2\ell_{2,\Delta}^2 = 3\ell_2^2 + \ell_1^2 \qquad [\text{direct calculation}]$$

COUNTING TRIPODS (7)

BACK to our counting problem: lift our tripod from $\mathbb{C}/\mathbb{Z}[i]$ to $\mathbb{C}$, and consider the triangle formed by its endpts. By translating, we move the largest angle of this triangle to 0, and let z, w be the other vertices (w/ $\arg(z) < \arg(w)$, say).

- For the largest angle of $\Delta(0\,z\,w)$ to be @ 0, we need it to be opposite the largest side, so $|z-w| \geq |z|, |w|$.
- For the angle θ @ 0 to have $\theta \leq 2\pi/3$, note that

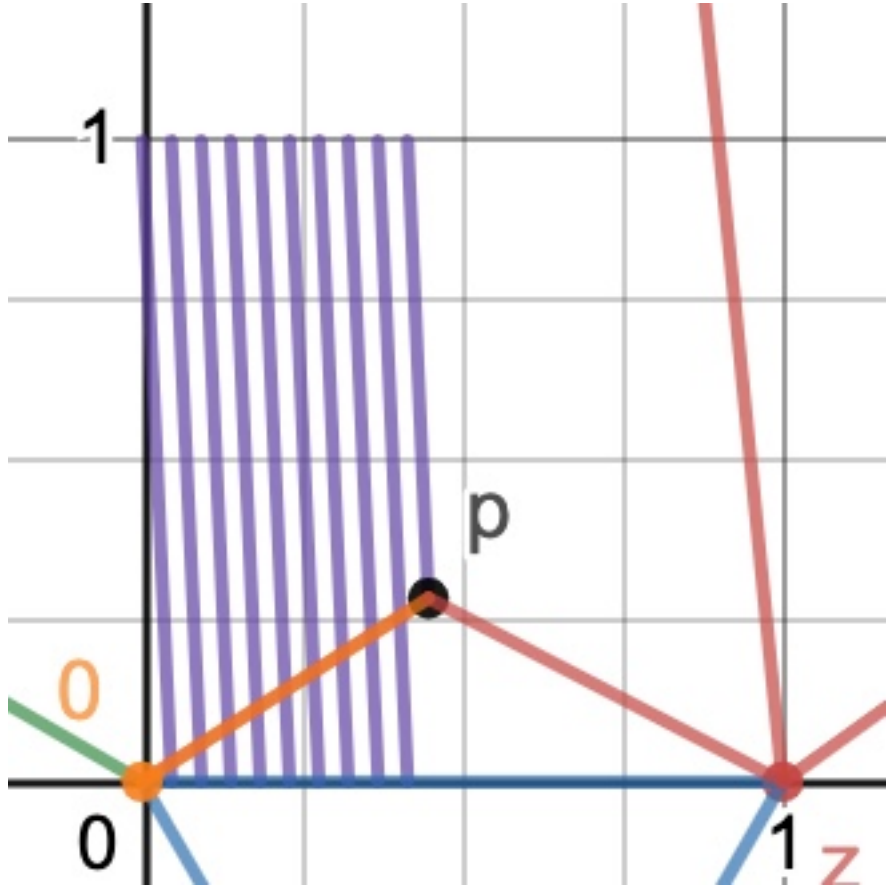
$$|z-w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\theta, \text{ and } \cos\theta \text{ on } [0,\pi],$$

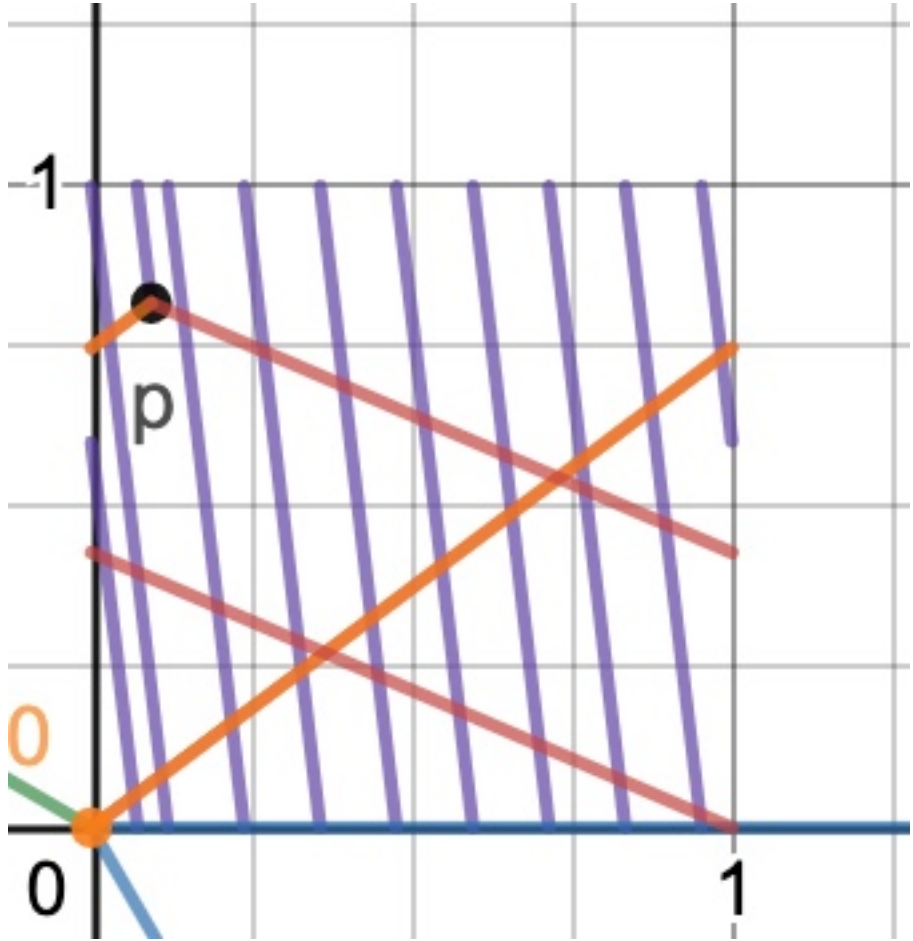
so $\theta \leq 2\pi/3 \iff \cos\theta \geq -1/2$, so want

$$\cos\theta = \frac{|z|^2+|w|^2-|z-w|^2}{2|z||w|} \geq -\frac{1}{2} \iff |z|^2+|w|^2+|z||w| \geq |z-w|^2$$

$$\iff (|z|+|w|)^2 \geq |z-w|^2 + |z||w|.$$

- & we want $\ell(\succ(z,w)) = |ze^{i\pi/3} + we^{-i\pi/3}| \leq R$.


1
p
0
0
1
z


1
p
0
0
1

COUNTING TRIPODS ⑧

So our counting problem becomes:

$$N_{\succ}(R) = \#\Big\{ (z,w) \in \mathbb{Z}[i] : \begin{array}{l} (0)\ z = x+iy \quad w = u+iv \quad \gcd(x,y,u,v) = 1. \\ (1)\ |z|^2 + |w|^2 + |z||w| \geq |z-w|^2 \geq |z|^2, |w|^2 \\ (2)\ |ze^{i\pi/3} + we^{-i\pi/3}| \leq R \end{array} \Big\}.$$

From standard lattice point counting results, we will have

$$N_{\succ}(R) \cong \frac{1}{\zeta(4)} R^4 \cdot \mathrm{vol}\Big((\rho, w) \in \mathbb{C}^2 : \begin{array}{l} (1)\ |\rho|^2 + |w|^2 + |\rho||w| \geq |\rho - w|^2 \geq |\rho|^2, |w|^2 \\ (2)\ |\rho e^{i\pi/3} + we^{-i\pi/3}| \leq 1 \end{array} \Big)$$

We now turn to computing this volume, call it $\mathrm{vol}(\Omega^{\succ})$

We will understand this set by fixing ρ and considering the set of w in the fiber over ρ, that is, the set

$$\Omega^{\succ}_{\rho} = \Big\{ w \in \mathbb{C} : \begin{array}{l} (1)\ |\rho|^2 + |w|^2 + |\rho||w| \geq |\rho - w|^2 \geq |\rho|^2, |w|^2 \\ (2)\ |\rho e^{i\pi/3} + we^{-i\pi/3}| \leq 1 \end{array} \Big\}$$

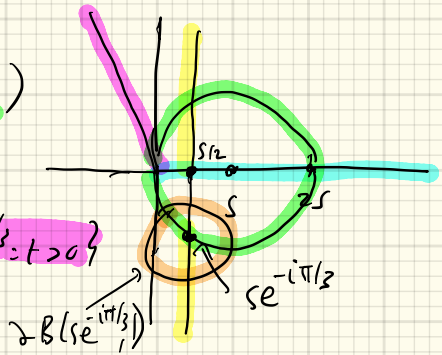
- Note that the $\mathrm{vol}(\Omega^{\succ}_{\rho})$ only depends on $|\rho|$

COUNTING TRIPODS (9)

Since $\mathrm{vol}(\Omega^{\lambda}_{\rho})$ only depends on $|\rho|$, we consider Ω^{λ}_{s}, $s>0$

$$\left[\ \Omega^{\lambda}_{\rho e^{i\theta}} = e^{i\theta}\,\Omega^{\lambda}_{\rho}\ \right]$$

Putting $\rho = s > 0$ $(s \in \mathbb{R})$ means



(a) $|\rho - w| \geq |\rho| \Rightarrow w \notin B(s, s)$

(b) we want w in the region between positive x-axis & ray $\{te^{2\pi i/3} : t>0\}$

(c) $|\rho - w| \geq |w|$, say $w = x+iy$

$$(x-s)^2 + y^2 \geq x^2+y^2 \iff s^2 - 2sx \leq 0 \iff x \leq s/2$$

(d) finally, $|\rho e^{\pi i/3} + we^{-i\pi/3}| \leq 1$

becomes $|w + se^{2\pi i/3}| \leq 1 \iff |w - se^{-i\pi/3}| \leq 1.$

The above illustration shows a region w/ $\Omega^{\lambda}_{s} = \emptyset$, since there is no intersection.

COUNTING TRIPODS

$$\Omega_s^{\lambda} = \left\{ w \in \mathbb{C} : \begin{array}{l} w \notin B(s,s),\ \operatorname{Re} w \le s/2,\ 0 \le \arg(w) \le 2\pi/3, \\ w \in B(se^{-i\pi/3}, 1) \end{array} \right\}.$$

Claims: (1) $\Omega_s^{\lambda} = \emptyset \quad \forall\, s > 1$

(2) The region Ω_s^{λ} changes shape @ $s = 1/\sqrt{3}$, when the circles $|w - s| = s$, $|w - se^{-i\pi/3}| = 1$ & the line $\operatorname{Re} w = s/2$ intersect @ $w_0 = se^{i\pi/3} = \frac{1}{\sqrt{3}} e^{i\pi/3}$

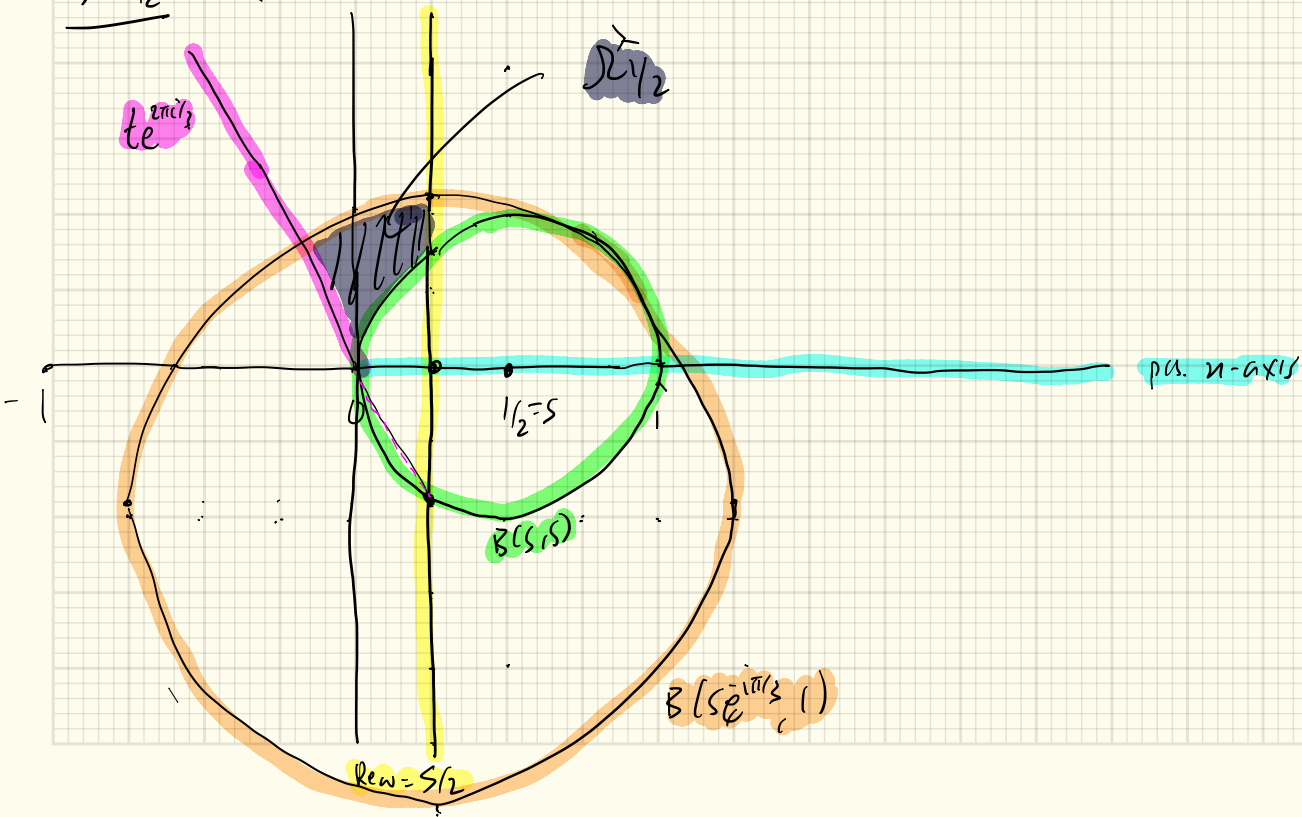
$$\left[\begin{array}{l} \left| \frac{1}{\sqrt{3}} e^{i\pi/3} - \frac{1}{\sqrt{3}} \right| = \left| -\frac{1}{2\sqrt{3}} + \frac{1}{2} i \right| = \sqrt{\frac{1}{12} + \frac{1}{4}} = \frac{1}{\sqrt{3}} \\ \left| \frac{1}{\sqrt{3}} e^{i\pi/3} - \frac{1}{\sqrt{3}} e^{i\pi/3} \right| = \frac{1}{\sqrt{3}} |\sqrt{3} i| = 1, \quad \operatorname{Re}(w_0) = \frac{1}{2\sqrt{3}} = \frac{(1/\sqrt{3})}{2} \end{array} \right]$$

(3) Note also that the line $\{ te^{i2\pi/3} : t \in \mathbb{R} \}$ passes through the point $se^{-i\pi/3}$ $(= -se^{2\pi i/3})$

COUNTING TRIPODS (11)

Here are some pictures of the region $\hat{\Omega}_s$

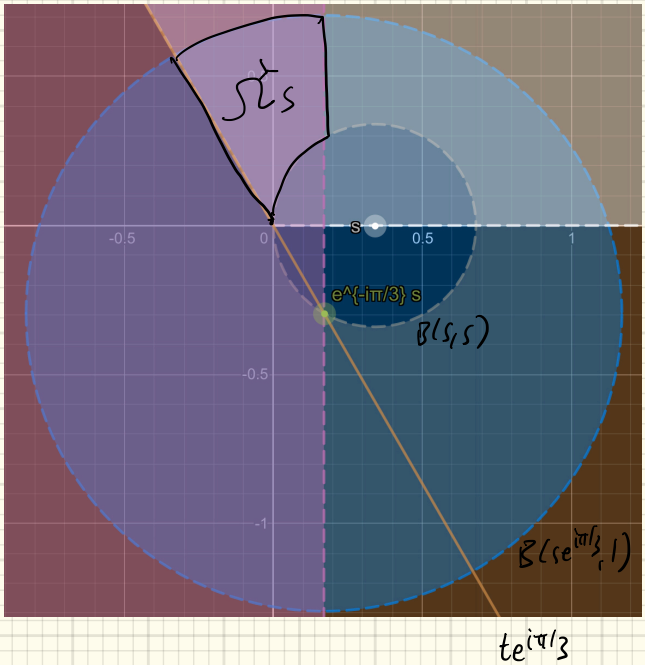
$s = 1/2$: (hand-drawn)



COUNTING TRIPODS (12)

Here are some pictures of the region Ω_s^*

$s = 0.34$

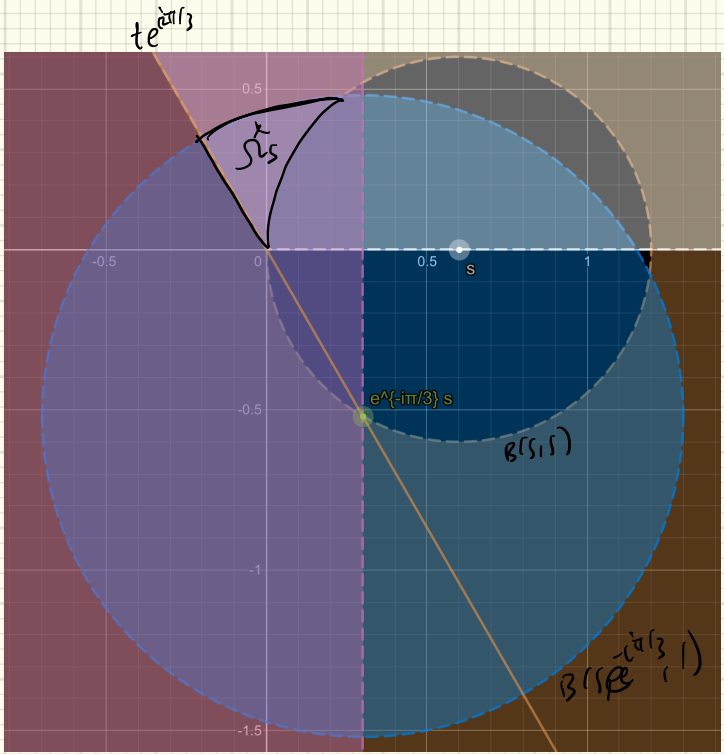


$(s < 1/\sqrt{3})$

COUNTING TRIPODS (B)

Here are some pictures of the region Ω_s^1

$s = 0.6$



$[s > 1/\sqrt{3}]$,

note change in shape of region.

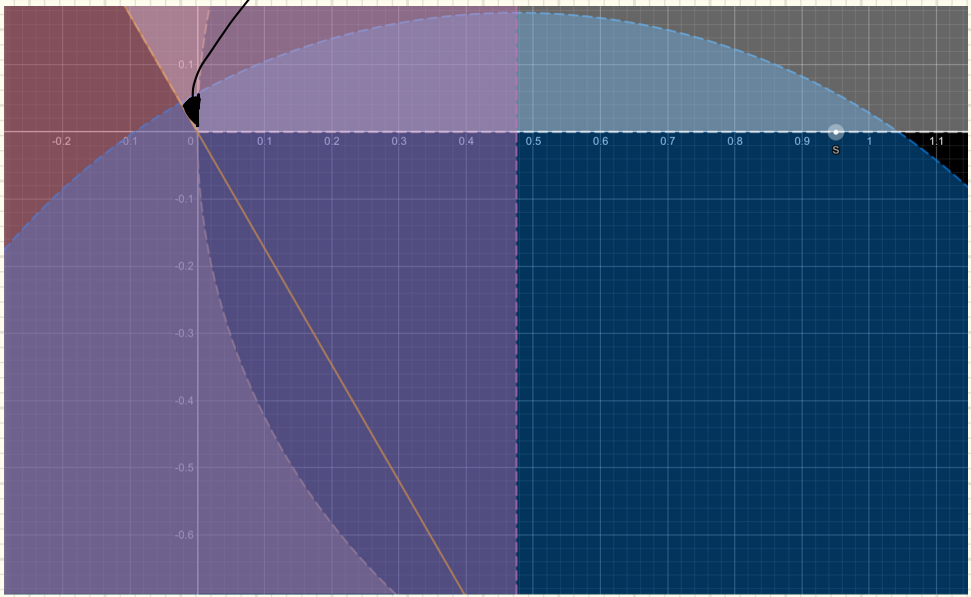
COUNTING TRIPODS

Here are some pictures of the region Ω_s^λ

$s = 0.95$

Ω_s^λ

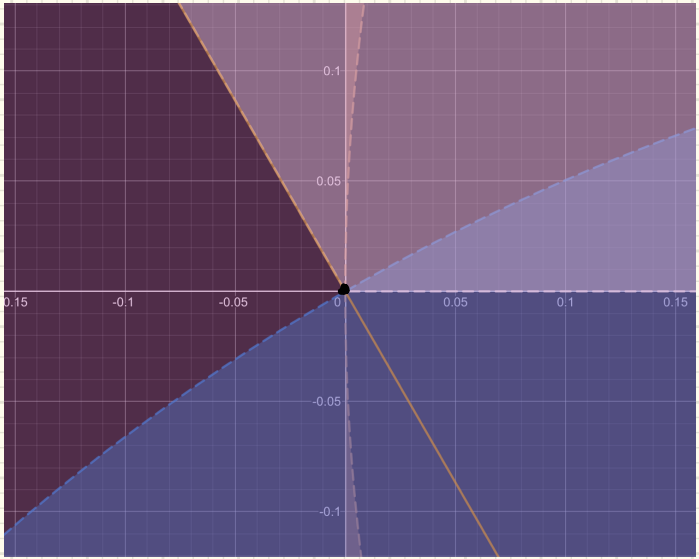
Note Ω_s^λ is very small for s close to 1.



COUNTING TRIPODS

Here are some pictures of the region Ω_s^λ

$s=1$ The region degenerates to the pt. $\{c\}$.



COUNTING TRIPODS (16)

Let $V(s) = \mathrm{Vol}(\Omega_s^\lambda)$.

Then $\mathrm{Vol}(\Omega^\lambda) = \int_0^{2\pi}\int_0^1 V(s)\, s\, dr\, d\theta = 2\pi \int_0^1 s V(s)\, ds.$

<u>Claim</u>

$$V(s) = \begin{cases} \int_{\pi/6}^{\pi/3}\int_{2s\cos\theta}^{1} r\, dr\, d\theta = \frac{\pi}{12}(1-2s^2), & 0<s<1/\sqrt{3} \\ \int_{\cos^{-1}(1/2s)}^{\pi/3}\int_{2s\cos\theta}^{1} r\, dr\, d\theta, & \frac{1}{\sqrt{3}}<s<1 \end{cases}$$

<u>Pf.</u> The key idea in this proof is to use polar coordinates <u>based</u> @ $se^{-i\pi/3}$, w/ $\theta = 0 \longleftrightarrow$ line passing through s.

Then $B(se^{-i\pi/3}, 1) \longleftrightarrow r \le 1$

$B(s, s) \longleftrightarrow r \geq 2s\cos\theta$

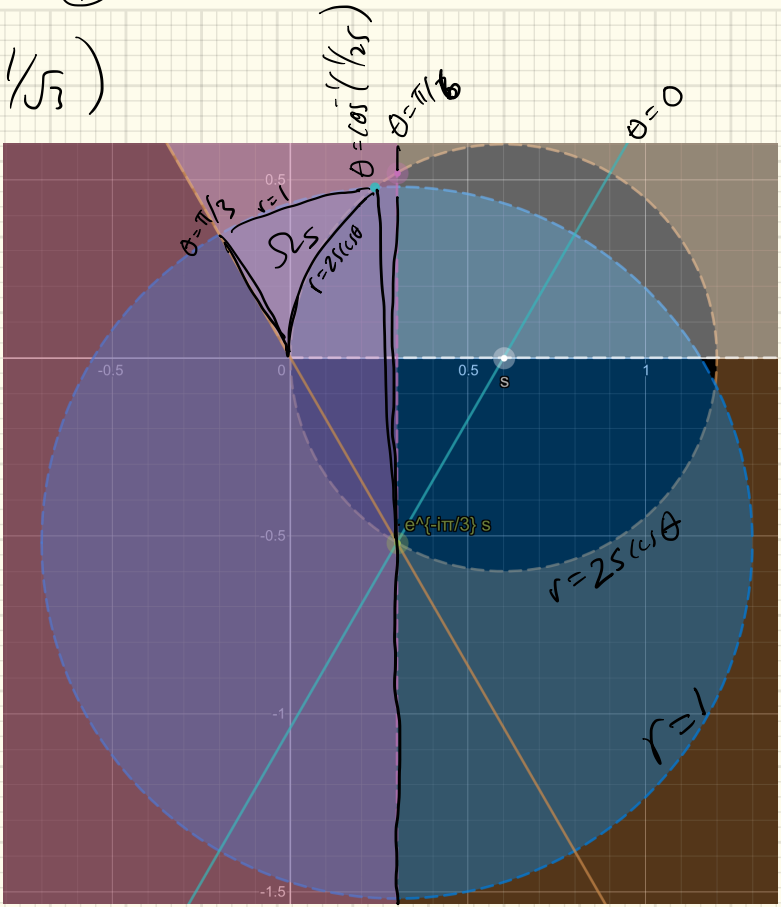
$\mathrm{Re}\, w = s/2 \longleftrightarrow \theta = \pi/6$

$\arg(w) = 2\pi/3 \longleftrightarrow \theta = \pi/3.$

COUNTING TRIPODS (17)

$s = 0.6 \quad (s > 1/\sqrt{3})$

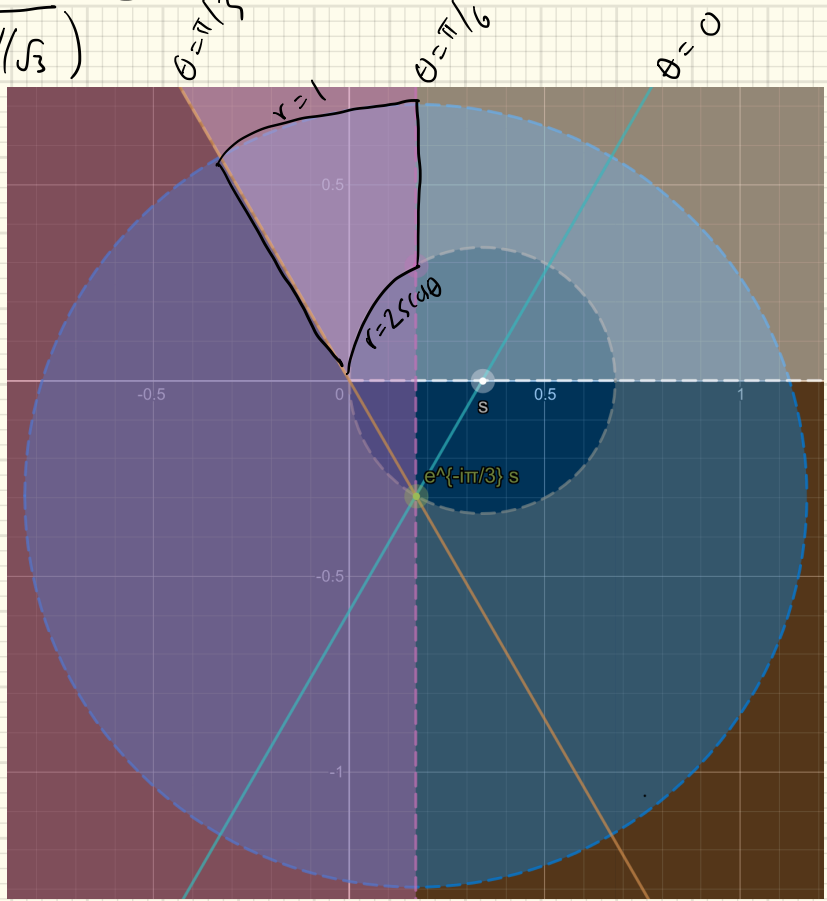
$$V(s) = \int_{\cos^{-1}(1/2s)}^{\pi/3} \int_{2s\cos\theta}^{1} r \, dr \, d\theta$$



COUNTING TRIPODS (18)

$s = 0.34$ $\left(s < 1/\sqrt{3}\right)$

$$V(s) = \int_{\pi/6}^{\pi/3} \int_{2s\cos\theta}^{1} r \, dr \, d\theta$$



So we can see that

$$\Omega_s^{\lambda} = \left\{ se^{-i\pi/3} + re^{i\theta} : \begin{array}{l} 2s\cos\theta \le r \le 1 \\ \max(\cos^{-1}(\frac{1}{2s}), \pi/6) \le \theta \le \pi/3 \end{array} \right\}.$$

So $$V(s) = \int_{\max(\cos^{-1}(1/2s), \pi/6)}^{\pi/3} \int_{2s\cos\theta}^{1} r \, dr \, d\theta$$

$$= \frac{1}{2} \int_{\max(\cos^{-1}(1/2s), \pi/6)}^{\pi/3} \left(1 - 4s^2\cos^2\theta \right) d\theta$$

<u>Note that</u> for $s_0 = 1/\sqrt{3}$, $\cos^{-1}(1/2s_0) = \cos^{-1}(\sqrt{3}/2) = \pi/6$.

We'll now compute these integrals.

COUNTING TRIPODS

Claim

$$V(s) = \begin{cases} \frac{\pi}{12}(1-2s^2) & 0 < s < 1/\sqrt{3} \\ \frac{1}{2}\left[\frac{\pi}{3} + 2s^2\cos^{-1}\left(\frac{1}{2s}\right) - 1\cdot\cos^{-1}\left(\frac{1}{2s}\right) - \left(2\pi/3 + \sqrt{3}/2\right)s^2 + \sqrt{s^2 - 1/4}\right] & \frac{1}{\sqrt{3}} < s < 1 \end{cases}$$

Pf. For $0 < s < 1/\sqrt{3}$

$$V(s) = \int_{\pi/6}^{\pi/3} \frac{1}{2}\left(1 - 4s^2\cos^2\theta\right) d\theta = \frac{1}{2}\left[\left(\pi/3 - \pi/6\right) - 4s^2 \int_{\pi/6}^{\pi/3} \cos^2\theta \, d\theta\right]$$

$$\int_{\pi/6}^{\pi/3} \cos^2\theta \, d\theta = \pi/12$$

$$= \frac{1}{2}\left[\left(\pi/3 - \pi/6\right) - 4s^2 \cdot \pi/12\right]$$

$$= \pi/12\left[1 - 2s^2\right]$$

COUNTING TRIPODS

Claim

$$V(s) = \begin{cases} \frac{\pi}{12}(1-2s^2) & 0 < s < 1/\sqrt{3} \\ \frac{1}{2}\left[\frac{\pi}{3} + 2s^2\cos^{-1}\left(\frac{1}{2s}\right) - \cos^{-1}\left(\frac{1}{2s}\right) - \left(2\pi/3 + \sqrt{3}/2\right)s^2 + \sqrt{s^2 - 1/4}\right] & \frac{1}{\sqrt{3}} < s < 1 \end{cases}$$

Pf. $\frac{1}{\sqrt{3}} < s < 1$,

$$V(s) = \int_{\cos^{-1}(1/2s)}^{\pi/3} \frac{1}{2}\left(1 - 4s^2\cos^2\theta\right)$$

$$= \frac{1}{2}\left[\pi/3 - \cos^{-1}(1/2s)\right] - \frac{1}{2}\cdot 4s^2 \cdot \int_{\cos^{-1}(1/2s)}^{\pi/3} \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\left[\pi/3 - \cos^{-1}(1/2s)\right] - \frac{1}{2}4s^2 \cdot \left(\theta/2 + \frac{1}{4}\sin(2\theta)\right)\Big|_{\cos^{-1}(1/2s)}^{\pi/3}$$

$$= \frac{1}{2}\left[\pi/3 - \cos^{-1}\left(\frac{1}{2s}\right) - 4s^2\left(\left(\pi/6 - \frac{\cos^{-1}(1/2s)}{2}\right) + \frac{1}{4}\left(\frac{\sqrt{3}}{2} - \frac{1}{s}\sqrt{1 - 1/4s^2}\right)\right)\right]$$

$$\int \cos^2\theta \, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) + C$$

$$\sin\left(\cos^{-1}(1/2s)\right) = \sqrt{1 - \frac{1}{4s^2}}$$

$$\cos\left(\cos^{-1}(1/2s)\right) = 1/2s$$

$$\sin\left(2\cos^{-1}(1/2s)\right) = 2\sin\left(\cos^{-1}\left(\frac{1}{2s}\right)\right)\cdot\cos\left(\cos^{-1}(1/2s)\right) = \frac{1}{s}\sqrt{1 - 1/4s^2}$$

COUNTING TRIPODS

Claim

$$V(s) = \begin{cases} \frac{\pi}{12}(1-2s^2) & 0 < s < 1/\sqrt{3} \\ \frac{1}{2}\left[\frac{\pi}{3} + 2s^2\cos^{-1}\left(\frac{1}{2s}\right) - \cos^{-1}\left(\frac{1}{2s}\right) - \left(2\pi/3 + \sqrt{3}/2\right)s^2 + \sqrt{s^2 - 1/4}\right] & \frac{1}{\sqrt{3}} < s < 1 \end{cases}$$

Pf. We've seen, for $\frac{1}{\sqrt{3}} < s < 1$,

$$V(s) = \frac{1}{2}\left[\pi/3 - \cos^{-1}\left(\frac{1}{2s}\right) - 4s^2\left(\pi/6 - \frac{\cos^{-1}(1/2s)}{2}\right) + \frac{1}{4}\left(\frac{\sqrt{3}}{2} - \frac{1}{s}\sqrt{1 - 1/4s^2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\pi/3 + 2s^2\cos^{-1}(1/2s) - \cos^{-1}(1/2s) - (2\pi/3 + \sqrt{3}/2)s^2 + \sqrt{s^2 - 1/4}\right]$$

$\square$

Now we are ready to compute $\mathrm{Vol}(\Omega) = 2\pi \int_0^1 s\,V(s)\,ds$.

We will break it up into two pieces:

(A) $$\int_0^{1/\sqrt{3}} s\,V(s)\,ds = \int_0^{1/\sqrt{3}} \pi/12\, s(1-2s^2)\,ds = \pi/12 \left[s^2/2 - s^4/2 \right]_0^{1/\sqrt{3}}$$

$$= \pi/24 \left[1/3 - 1/9 \right] = \frac{2\pi}{9\cdot 24} = \pi/9\cdot 12 = \pi/108.$$

(B) $$\int_{1/\sqrt{3}}^1 s\,V(s)\,ds = \int_{1/\sqrt{3}}^1 s \left[\frac{1}{2} \left[\frac{\pi}{3} + 2s^2 \cos^{-1}\left(\frac{1}{2s}\right) - \cos^{-1}\left(\frac{1}{2s}\right) - \left(2\pi/3 + \sqrt{3}/2\right) s^2 + \sqrt{s^2 - 1/4} \right] \right] ds$$

We break this up into:

(1) $\frac{1}{2} \int_{1/\sqrt{3}}^1 \pi s/3\, ds$ (2) $\frac{1}{2} \int_{1/\sqrt{3}}^1 (2s^3 - s) \cos^{-1}(1/2s)\, ds$

(3) $-\frac{1}{2} \int_{1/\sqrt{3}}^1 \left(2\pi/3 + \sqrt{3}/2\right) s^3\, ds$ (4) $\frac{1}{2} \int_{1/\sqrt{3}}^1 s \sqrt{s^2 - 1/4}\, ds$

We'll evaluate each of these in turn.

COUNTING TRIPODS (24)

(1) $\frac{1}{2}\int_{1/\sqrt{3}}^{1} \pi s/3 \, ds = \frac{\pi}{12} s^2 \Big|_{1/\sqrt{3}}^{1} = \frac{\pi}{12}\left(1-\frac{1}{3}\right) = \pi/18$

(2) $\frac{1}{2}\int_{1/\sqrt{3}}^{1} (2s^3 - s)\cos^{-1}(1/2s) = \frac{1}{2}\left[\frac{1}{216}(7\sqrt{3}+4\pi)\right]$ (via Wolfram Alpha)

$$\int_{\frac{1}{\sqrt{3}}}^{1} \left(-s+2s^3\right)\cos^{-1}\left(\frac{1}{2s}\right) ds = \frac{1}{432}\left(7\sqrt{3}+4\pi\right) \approx 0.057154$$

$$\frac{1}{2}\int \left(-s+2s^3\right)\cos^{-1}\left(\frac{1}{2s}\right) ds = \frac{1}{96} s\left(\sqrt{4-\frac{1}{s^2}}\left(5-2s^2\right)+24s\left(s^2-1\right)\cos^{-1}\left(\frac{1}{2s}\right)\right) + \text{constant}$$

(3) $-\frac{1}{2}\int_{1/\sqrt{3}}^{1} \left(2\pi/3 + \sqrt{3}/2\right) s^3 = -\frac{1}{2}\left(2\pi/3 + \sqrt{3}/2\right)\cdot s^4/4 \Big|_{1/\sqrt{3}}^{1}$

$= -\frac{1}{2}\left(2\pi/3+\sqrt{3}/2\right)\cdot\frac{1}{4}\left(1-1/9\right) = -1/9\left(2\pi/3+\sqrt{3}/2\right) = -2\pi/27 - \sqrt{3}/18$

(4) $\frac{1}{2}\int_{1/\sqrt{3}}^{1} s\sqrt{s^2-1/4}\, ds \quad u = s^2 - 1/4 \quad \frac{du}{2} = ds$

$= \frac{1}{4}\int_{1/12}^{3/4} \sqrt{u}\, du = \frac{1}{4}\cdot\frac{2}{3} u^{3/2}\Big|_{1/12}^{3/4} = \frac{1}{6}\left[(3/4)^{3/2} - (1/12)^{3/2}\right]$

$= 1/6\left[3\sqrt{3}/8 - 1/24\sqrt{3}\right] = 1/6\left[\frac{3\sqrt{3}\cdot 3\sqrt{3} - 1}{24\sqrt{3}}\right] = \frac{1}{6}\cdot\frac{26}{24\sqrt{3}} = \frac{13}{72\sqrt{3}}$

so $\int_0^1 sV(s)\,ds = \int_0^{1/\sqrt{3}} sV(s)\,ds + \int_{1/\sqrt{3}}^1 sV(s)\,ds$

$$= \text{A} + \text{1} + \text{2} + \text{3} + \text{4}$$

$$= \pi/108 + \pi/18 + \tfrac{1}{2}\left[\tfrac{1}{216}(7\sqrt{3} + 4\pi)\right] - 2\pi/27 - \sqrt{3}/18 + 13/72\sqrt{3}$$

$$= \pi\left(\frac{1}{108} + \frac{1}{18} + \frac{4}{432} - 2/27\right) + \sqrt{3}\left[\frac{7}{432} - \frac{1}{18} + \frac{13}{216}\right] \qquad \left(\frac{13}{72\sqrt{3}} = \sqrt{3}\cdot\frac{13}{216}\right)$$

$$= \pi\left(\frac{1}{54} + 1/18 - 2/27\right) + \sqrt{3}\left[\frac{7 - 24 + 26}{432}\right]$$

$$= \pi\left(\frac{1+3-4}{54}\right) + \sqrt{3}\left[\frac{9}{432}\right] = \sqrt{3}/48.$$

So we have proved:

<u>Lemma</u> $\mathrm{Vol}(\Omega^{\lambda}) = \sqrt{3}/48 \cdot 2\pi$ $\square$

$$= \sqrt{3}/24 \cdot \pi$$

Now, $N_\lambda(R) \cong \frac{1}{\zeta(4)} \mathrm{Vol}(\Omega^\lambda) \cdot R^4$

$$= \frac{90}{\pi^4} \cdot \frac{\sqrt{3}}{24} \cdot \pi = \frac{15\sqrt{3}}{4\pi^3}$$

The lattice point counting comes from $\Omega^\lambda_R = R \cdot \Omega^\lambda$, & in general, for "nice" regions $\Omega \subset \mathbb{C}^2 \cong \mathbb{R}^4$,

$$\# \mathbb{Z}^4_{\text{prim}} \cap R\Omega \cong \frac{\mathrm{vol}(\Omega) \cdot R^4}{\zeta(4)}.$$

ANDY'S EMAIL (5/26/20)

Reminder: the idea here was that the counting problem we want to study might become easier when it is formulated not on a surface but on an object of higher dimension. If we think of the usual "saddle connections of quadratic differentials" story as having to do with the group SL(2), then the idea is that the analogous story for SL(N) will live naturally on an object of complex dimension N-1.

So e.g. for SL(3) we want to have complex dimension 2. This 2-dimensional object is supposed to be built from flat patches that look like C^2, with the structure group being C^2 (translations) twisted by S_3 (permuting the triple x,y,-x-y) -- in a way analogous to how a flat surface could be built from patches that look like C, with the structure group C (translations) twisted by S_2 (permuting x,-x).

On this 2-complex-dimensional object the things we want to count should appear as real-codimension-1 webs. In particular if you take any 1-complex-dimensional slice through the object you will meet a counting problem of the usual sort -- saddle connections, tripods, maybe bigger webs. But the problem is supposed to be easier to get a grip on once you use the higher-dimensional picture. In particular, there ought to be an SL(2,R) action on the moduli space of these higher-dimensional objects, which preserves the webs.

The new progress is that I think I understand a general scheme for constructing these higher-dimensional objects, and I worked it out in a baby case. In this baby case, one slice is the complex plane equipped with the quadratic differential \phi_2 = 0, and the cubic differential \phi_3 = (y^2 - 1) dy^3. (Or you could think of it as CP^1 with a meromorphic cubic differential, with a pole at z=infinity.) So on this slice we can write the spectral curve

x^3 + y^2 - 1 = 0

and the flat coordinates around a given point y are period integrals of the form

\int x dy

where the integral runs over an arc on the spectral curve, whose initial and final points are two of the preimages (x,y) of y.

I think that in this case the desired higher-dimensional space is just C^2, with two coordinates (y1,y2), and the local flat coordinates are given in a slightly more complicated fashion: you write down a family of spectral curves as

x^3 + (y1 + y2 x)^2 - 1 = 0

and then the local flat coordinates are integrals of the form

\int x dy1 + 1/2 x^2 dy2

(nb: this form is indeed closed, so the integrals only depend on the homology class of the path)

This kind of baby example is analogous to the case of an infinite-area surface in the SL(2) case, and it does not exhibit the most interesting features like growth of the number of webs with length -- in this baby example, if I'm not confused, there are only finitely many webs, no matter what length you take.
But it is still sort of an interesting toy to play with. e.g. you can draw the spectral networks corresponding to any fixed choice of y_2, and you find that the finite webs always occur at the same phase, independent of y_2 -- as you expect if they are really slices through a single web of real codimension 1.

I didn't try to think about how SL(2,R) transforms this example. I guess that kind of action should still exist even in this "infinite area" setting, right? (But in this case we don't expect an invariant measure?)

ANDY'S EMAILS 5/26/20

(relationship to Fock-Thomas)

I think it ought to be related to the higher complex structures -- a lot of the ideas are the same. The natural guess would be that the moduli space parameterizing these "higher flat spaces" is a vector bundle over the space of higher complex structures.

I have some *very rough* idea of how that could go. The basic way we construct one of these "higher flat spaces" is to take moduli of holomorphic Lagrangian submanifolds inside of a holomorphic symplectic manifold X, up to an equivalence relation. The basic case is to take X=T*C, the holomorphic cotangent bundle. But you can also take a deformation of T*C (still diffeomorphic but different as a complex maifold). Again you have to take these deformations modulo some equivalence relation -- lots of different X are going to lead to the same higher flat space. Fock-Thomas's notion of "higher complex structure on C" looks (if you squint optimistically) a little bit like what you might need in order to construct a deformation of T*C, up to some equivalence relation which morally keeps track only of something like an N-th infinitesimal neighborhood of the zero section.

HIGHER COMPLEX STRUCTURES

V.V. FOCK, A. THOMAS

Abstract. We introduce and analyze a new geometric structure on topological surfaces generalizing the complex structure. To define this so called higher complex structure we use the punctual Hilbert scheme of the plane. The moduli space of higher complex structures is defined and is shown to be a generalization of the classical Teichmüller space. We give arguments for the conjectural isomorphism between the moduli space of higher complex structures and Hitchin's component.

Introduction

Poincaré's uniformization theorem links complex structures on a surface Σ to homomorphisms from the fundamental group of the surface to $PSL(2,\mathbb{R})$, the automorphism group of the hyperbolic plane. With this, Teichmüller space $\mathcal{T}_\Sigma$ can be identified with the connected component of the character variety of faithful and discrete representations of the fundamental group in $PSL(2,\mathbb{R})$:

$$\mathcal{T}_\Sigma \cong \mathrm{Hom}^{\mathrm{discrete}}(\pi_1(\Sigma), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}).$$

In his celebrated paper [Hi92], Nigel Hitchin proves the existence of a connected component of the character variety for an adjoint group of a split real form of any complex simple Lie group (for instance $PSL(n,\mathbb{R})$) consisting of faithful and discrete representations and parametrized by holomorphic differentials. These components generalize Teichmüller space. His methods are analytic, using the theory of Higgs bundles. Teichmüller space has also geometric descriptions: it is the moduli space of hyperbolic structures (metrics with constant curvature -1) and also the moduli space of complex structures. The natural question is then if there is a geometric description of Hitchin's component.

In this paper, we describe and analyze a new geometric structure on surfaces generalizing the complex structure. Conjecturally, the moduli space of the so called higher complex structure is isomorphic to Hitchin's component. This would give a purely geometric approach to higher Teichmüller theory. We give arguments in favor of the isomorphism.

The search for a geometric origin of Hitchin's component is of course not new. Goldman, Guichard-Wienhard, Labourie and others describe Hitchin's component via geometric structures on bundles over the surface. For $PSL(3,\mathbb{R})$, this geometric structure is the convex projective structure described by Goldman in [Go68]. For $n = 4$, Guichard and Wienhard describe convex foliated structure on the unit tangent bundle in [GW08]. Labourie introduces the concept of an Anosov representation in [La06]. The drawback of these constructions is that the bundle on which the geometric structure is defined is not canonically associated to the surface.

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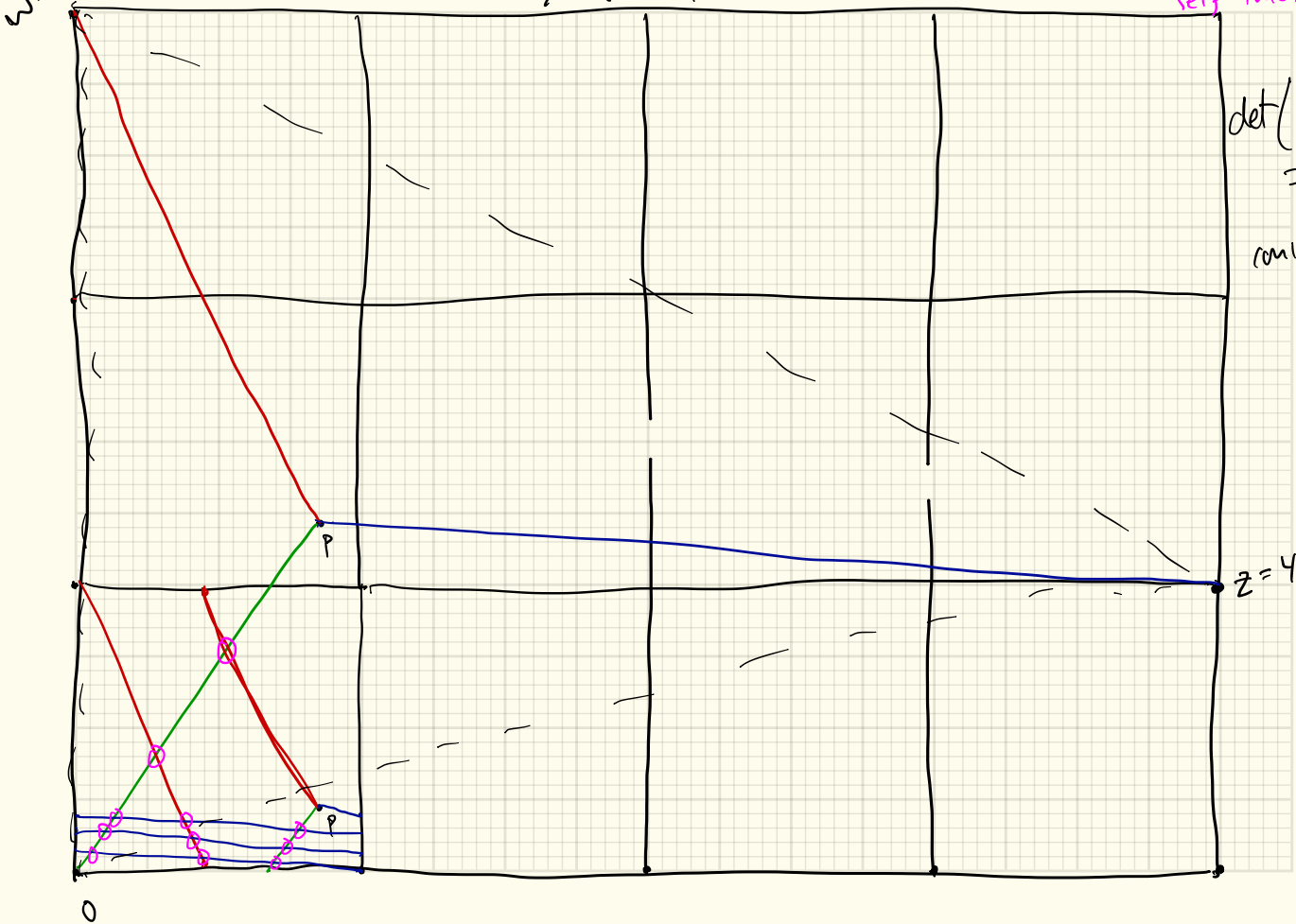
$\lambda(4+i, 3i)$

// transverse self-intersection.



$\det\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} = 11.$

coincidence?